Good Index Choosing for Polarized Relay Channel

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Abstract

The Polar coding is a method which have been proposed by Arikan and it is one of the first codes that achieve the capacity for vast numerous channels. This paper discusses relay channel polarization in order to achieve the capacity and it has been shown that polarization of two relay channels can be given a more achievable rate region in the general form. This method is compatible with the original vision of polarization based on the combining, splitting and polarizing of channels and it has been shown that the complexity of encoding and decoding for these codes in mentioned method are $O(N\, log\, N)$, and also error probability for them is $O(2^{-(N)^{\beta}})$. Choose the best sub-channels in polarized relay channels for sending data is a big trouble in this structure. In this paper, we have been presented a new scheme for choosing a good index for sending the information bits in relay channels polarized in order to have the best performance by using sending information bits over FIF sets.

Keywords: Relay channel; Polar code; Channel polarization; Capacity; Relay channel polarization; Good index of relay channel.

1. Introduction

The relay channel is a communication channel which has a sender and receiver assistant in communication by utilizing of a relay node [1]. Specifying a memoryless relay channel is can be given by the probability distribution $W(Y_r, Y|X, X_r)$. In the defined prbability, $, X_r$ are the symbol transmitted by the source and the symbol transmitted by the relay, respectively, are Y_r is the symbol received by the relay and finally Y is the symbol received by the destination. This defination has been illustrated generally in Fig. 1. In this model of relay channel, it has been assumed that the message M is uniformly distributed throughout the message set and the average probability of error is defined as

$$P_e^{(n)} = \Pr\{\hat{M} \neq M\} \tag{1}$$

The rate R is said to be achievable if there is a message with $(2^{NR}, N)$ codes property such that

$$\lim_{N \to \infty} P_e^{(n)} = 0 \quad . \tag{2}$$

In [2], it is well-known and it has been shown in [1], which the capacity of the relay channel in general form is still an open problem and this causes the inportance of relay channel study.

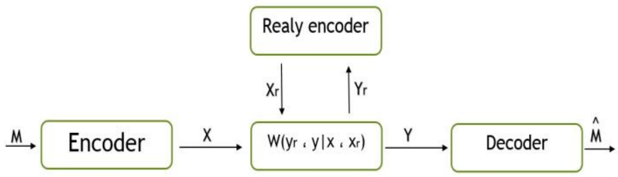

Fig. 1. A typical relay channel [1]

The cut-set bound is the outer bound of the capacity for the relay channel and it is established in [2] as follows:

$$C \leq \max_{p(x,x_r)} \min \{I(X, X_r; Y_r), I(X; Y, Y_r | X_r)\}. \tag{3}$$

Decode-and-forward (DF) and compress-and-forward (CF) are the main coding scenarios for information transmitting in relay channels. In DF strategy, after recoverying the transmitted message from sender by relay, the relay forward it to the destination and this information helps the receiver to complete the best observation of the main link. Lower bound of DF is given by [2]:

$$C \geq R_{DF} := \max_{p(x,x_r)} \min \{I(X, X_r; Y_r), I(X; Y_r | X_r)\}, \tag{4}$$

Recently, polar codes, introduced by Arikan, give a way that can be called channel polarization technique and this scheme have been extended to various multi-terminal scenarios, such as the Multiple-Access Channels [5-7], Broadcast Channels [8]-[9] as well. These codes are one of the first codes that can achieve the capacity for binary input symmetric channels [3]-[4].

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In this paper, firstly, the well-known channel polarization phenomenon has been presented for relay chnnel and has been shown that the polarization of cut-set bound in relay channels how can be used effectively. The proposed schemes have the same standard properties of a typical polar codes with respect to encoding and decoding with the complexity $O(N.logN)$. The scaling of the block error probability is an exponential function of the block length, which decay like $(2^{-(N)^{\beta}})$, where $0 \leq \beta \leq 0.5$. The choosing the best sub-channels and good index in polarized relay channels for sending the data is a big trouble problem. In our structure, it has been solved by using sending the information over an FIF set in indices at Section V.

This paper is organized as follows: In Section II, the backbone material of a typical polar code and relevant previous works on relay channels have been reviewed. It has been shown also that the DF and CF strategy in general three terminal relay channel with using polar codes are achievable in Section III. In Section IV, polarization of relay channel for cut-set bound has been shown. In Section V, we introduce a new scenario of choosing good indices for relay channels polarization and finally. Last but not the least, at Section VI, we conclude the paper.

2. polar codes and relay channels

In this section, we supply a brief overview of the main work of Arikan [3] for single link polar codes technique and channel polarization method we present a brief part of previous works about the relay channels.

A. Polar codes

Constructing a typical polar codes is based upon a phenomenon that is well-known as channel polarization method [4]. The basic channel polarization method is given by a matrix, which is:

$$G_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{5}$$

The kronecker power of G for any $n \succ 1$ can be defined according to the iterative matrix formula such as

$$G_2^{\otimes n} = \begin{bmatrix} G_2^{\otimes(n-1)} & 0 \\ G_2^{\otimes(n-1)} & G_2^{\otimes(n-1)} \end{bmatrix} \tag{6}$$

with initial point as (5). Following [3], for a typical DMC, the channel splitting can be defined as mapping of

$$(W, W) \rightarrow (W^-, W^+). \tag{7}$$

The synthesized channels

$$W^-: F_2 \rightarrow Y^2 \tag{8}$$

and

$$W^+: F_2 \rightarrow F_2 \times Y^2 \tag{9}$$

are given as follows:

$$W^-(y_1^2 | u_1) = \sum_{u_2 \in \{0,1\}} \frac{1}{2} W(y_1 | u_1 \oplus u_2) W(y_2 | u_2), \tag{10}$$

$$W^+(y_1^2, u_1 | u_2) = \frac{1}{2} W(y_1 | u_1 \oplus u_2) W(y_2 | u_2). \tag{11}$$

W^- and W^+ are bad and good channel, respectively, in comparing to the original channel by using the channel splitting method. Arikan uses the Bhattacharyya parameter for a typical channel W, which is denoted by $Z(W)$, in order to measure how a good channel and bad channel can be classified. This parameter in general form is defined by:

$$Z(W) = \sum_{y \in Y} \sqrt{W(y|0) W(y|1)}. \tag{12}$$

Channels with Bhattacharyya parameter close to "0" are almost noise-free channel and channels with $Z(W)$ close to "1" are almost full-noisy channels [4]. The sub-set of noise-free channels $I_N(W)$ is defined according to below for any $0 \leq \beta \leq 0.5$:

$$I_N(W) := \{i \in [N] : Z(W_{2N}^{(i)}) \leq \frac{2^{-(N)^{\beta}}}{N}\}. \tag{13}$$

Where [*N*] is denoted the set of channels, which has less or equal to *N*. The channel polarization technique certify that the fraction of good binary input channel, approaches the symmetric capacity *I(W)* when *N*approaches to infinity [3]-[4]. *I(W)* is the quantity of mutual information and it can be defined as the channel capacity of *W*. By using the polarization theorem, one can find out that polar codes achieves the capacity [3]. Let set *A* is characterized as

$$A = \{i \in [N] : Z(W_N^i) \in [0, \delta]\}, \tag{14}$$

where $\delta \succ 0$. For any $0 \leq \beta \leq 0.5$, the error probability of polar codes is determined by block error-probability when the Successive Cancellation (SC) decoding used:

$$P_e = \sum_{i \in A} Z(W_N^{(i)}) = o(2^{-2^{n\beta}}). \tag{15}$$

B. Previous works on relay channels with respect to polar coding:

There are two ideas in using of polar codes. The main idea, which was shown by Arikan, is that the polar codes can achieve the capacity for a large case of channels and it is equal to say that the rate of such scheme approaches the capacity of the channel. Also, the main idea is still capacity achieving polar codes in most papers about the relay channel. In [10], a practical method for acheiving the capacity of symmetric physically degraded relay channels with binary input has been shown effectively. By concluding [11], one can find out about the achievability of DF bound by means of polar codes in a stochastically channel, which is degraded with binary symmetric relaying and orthogonal receivers. Also in [11], it has been shown that polar codes can be applied to CF relaying. In [12-13]. it has been discussed that by utilizing the polar codes, one can achieve the capacity of symmetric degraded relay channel. Also the problem of achievability for two other lower bounds by utilizing polar coding techniques has been shown in [14]. The topic of increasing the capacity for $N \to \infty$ has been studied in [5-7] and it has been shown that this the capacity of one of the polarized channels increase while the other decrease. In [5-6], a method for the polarization of the MAC has been illustrated. It has been shown that the polarization of a general Multiple Access Channel with a point-to-point channel can be done and one can achieve more rate region [7].

In second approach, one can design a channel polarization scheme when the number of channels increases, N and show that the capacity region increases too. Changing of different bound of relay channel, especially cut-set bound is a fantastic vision at this approach. In this paper, we use the first and the second methodology together in order to choose good index for the relay channel polarization method.

3. DF and cf relaying using polar codes

In [10]-[13], for degraded relay channel, polar coding schemes have been proposed on the following Markov chain as:

$$X \to (X_R, Y_R) \to Y \tag{16}$$

Also, a polar coding scheme can be applied for compress-forward and it has been proposed in relay channels with orthogonal receiver components as well in [11].

Theorem1: based on the first viewpoint, (17), (18) give the following relation:

$$\mathrm{I}(\mathrm{W}^-) + \mathrm{I}(\mathrm{W}^+) = 2\mathrm{I}(\mathrm{W}), \tag{17}$$

and

$$\mathrm{I}(\mathrm{W}^-) \leq \mathrm{I}(\mathrm{W}) \leq \mathrm{I}(\mathrm{W}^+). \tag{18}$$

Where W^- and W^+ are bad and good channels, respectively.

Proof: The proof is the same as [3], [4].

The mapping of

$$(\mathrm{W}, \mathrm{W}) \to (\mathrm{W}^-, \mathrm{W}^+) \tag{19}$$

has been called polarization (one level polarization). The same mapping can be applied to W^- and W^+ to get $\mathrm{W}^{--}, \mathrm{W}^{-+}, \mathrm{W}^{+-}$ and W^{++} (which is second level polarization). For any arbitrary number of levels, the same process can be continued in order to polarize W. The channel polarization theorem states that the binary input channel can be polarized as the code length N goes to infinity, it means that they can be set in two different sets, one set become noise-free and the other become very noisy. In this viewpoint instead of

$$\mathrm{R}(\mathrm{W}) \leq \mathrm{I}(\mathrm{W}), \tag{20}$$

by using (18) more rate can achieve, so:

$$I(W) \leq R(W^+) \leq I(W^+) \tag{21}$$

We call this vision as channel polarization in order to achieve more capacity.

In second viewpoint, since

$$\mathrm{I}(\mathrm{W}^+) \geq \mathrm{I}(\mathrm{W}), \tag{22}$$

we can find out a way, which

$$\mathrm{R}(\mathrm{W}^+) \geq \mathrm{R}(\mathrm{W}). \tag{23}$$

In polar coding, a channel, which can send bits without noise, uses information bit (I), and a channel, which can send bits noisy, uses frozen bit (F) or redundancy. Indeed, the polarization idea has been used to propose polar codes and a recursive process leads to efficient coding structures (encoding and decoding structures).

4. polarization for relay channel

CF strategy for relay channels based on orthogonal receiver components, $\mathrm{Y} = (\mathrm{Y}', \mathrm{Y}'')$, can be used and one can applied the well-known polar coding scheme as:

$$W(y_r, y|x, x_r) = W(y', y_r|x) W(y''|x_r) \tag{24}$$

[11]. This viewpoint can achieve the symmetric CF rate. The main results for this section is given in the following theorem.

Theorem 2. **(**Symmetric DF and CF relaying using polar code). For any transmission rate

$$R < R(DF) \tag{25}$$

and any fixed rate

$$R < R(CF), \tag{26}$$

one can find a polar codes with block error probability

$$P_e^{(n)} = Pr\{\widehat{M} \neq M\} \tag{27}$$

under SC decoding. This block error probability is bounded as

$$P_e \leq O\left(2^{-(N)^\beta}\right), \tag{28}$$

where $0 < \beta < \frac{1}{2}$ and the relay channel is stochastically degraded.

Proof. This Proof is like the proof of Theorem 1 in [10] and Theorem 2 in [13].

Since the main results have been surveyed, only the structure of polar codes, can be achieved for N relay channel and the complexity of encoding and decoding is $O\ (N.\log N)$ [10], [13].

The notion of channel polarization has been extended to relay channels, wherein a technique is described to polarize a given binary-input relay channel same as in [3]-[5]-[6]. We prove polarization of cut-set bound. It has been shown after polarization for two relay channels, the capacity of one relay increases while the other decreases and the capacity region of the relay changes. Two independent uses of the channel W results relay channel W^2 according to the polarization of Fig. 1, we have:

$$X \to (X_1, X_2), \quad (29)$$

$$X_r \to (X_{r1}, X_{r2}) \quad (30)$$

and

$$Y \to (Y_1, Y_2). \quad (31)$$

The cut-set bound in a channel W^2, is described by two following quantities:

$$I(X_1X_2, X_{r1}X_{r2}; Y_1Y_2) = 2I_1(W) \quad (32)$$

and

$$I(X_1X_2; Y_{r1}Y_{r2}; Y_1Y_2|X_{r1}X_{r2}) = 2I_2(W) \quad (33)$$

Also the cut-set bound is as follows:

$$R < \min\{I(X_1^2, X_{r1}^2; Y_1^2), I(X_1^2; Y_1^2, Y_{r1}^2|X_{r1}^2)\}, \quad (34)$$

and we have;

$$X_1^2 = U_1^2\, G_2 \quad (35)$$

and

$$X_{r1}^2 = V_1^2\, G_2. \quad (36)$$

Now we get:

$$2I_1(W) = I(U_1U_2, V_1V_2; Y_1Y_2) = I(U_1V_1; Y_1Y_2) + I(U_2V_2; Y_1Y_2, U_1V_1) = I_1(W^-) + I_1(W^+) \quad (37)$$

and we also have:

$$2I_2(W) = I(U_1U_2; Y_{r1}Y_{r2}, Y_1Y_2|V_1V_2) = I(U_1; Y_{r1}Y_{r2}, Y_1Y_2|V_1V_2) + I(U_2; Y_{r1}Y_{r2}, Y_1Y_2, U_1|V_1V_2) = I_2(W^-) + I_2(W^+) \quad (38)$$

In this way, the polarized bad relay channel is:

$$U_1 \times V_1 \to Y_1Y_2Y_{r1}Y_{r2} \quad (39)$$

and the capacity indicates with both quantities: $I_1(W^-)$ and $I_2(W^-)$. At the other hand, the polarized good relay channel is:

$$\mathrm{U_2 \times V_2 \to Y_1Y_2Y_{r1}Y_{r2}U_1|V_1V_2} \quad (40)$$

and the capacity indicates with both quantities: $I_1(W^+)$ and $I_2(W^+)$. Let 's define the channel

$$W: X \times X_r \to Y \times Y_r \quad (41)$$

as a relay channel with $\{0,1\}$ input alphabet. Now, we also define two other relay channels as

$$W^-: X \times X_r \to Y^2 \times {X_r}^2 \quad (42)$$

and

$$W^+: X \times X_r \to Y^2 \times {Y_r}^2 \times X \times X_r, \quad (43)$$

in which we have:

$$W^-(y_1^2, y_{r1}^2|u_1, v_1) = \sum_{u_2\in x, v_2\in x_r} \frac{1}{4} W(y_1, y_{r1}|u_1 \oplus u_2, v_1 \oplus v_2) W(y_2, y_{r2}|u_2, v_2) \quad (44)$$

and

$$W^+(y_1^2, y_{r1}^2, u_1, v_1|u_2, v_2) = \frac{1}{4} W(y_1, y_{r1}|u_1 \oplus u_2, v_1 \oplus v_2) W(y_2, y_{r2}|u_2, v_2); \quad (45)$$

Where W^- and W^+ correspond to bad and good relay channels, respectively. For capacity bound, we get:

$$I_i(W^-) \le I_i(W) \le I_i(W^+)\ ;\ \mathrm{i}=1,2 \quad (46)$$

Correspondingly, for R, we get:

$$R(W^-) < R(W) < R(W^+) \quad (47)$$

***Example 1*:** Assume the polarization of Fig. 1 and consider all links are BEC with erasure probability $\varepsilon_1 = 0.5$ except the source to relay which is a BEC channel with parameter erasure probability ε_2. Now, we investigate three mentioned scenarios, through polarization of two relay channels. For specific case, if the link between source to relay is polarized but the link between the relay to destination is not polarized, then, we have:

$$C_1^+ = 0.75 + \min\{0.75, 1-\varepsilon_2\}$$

and

$$C_1^- = 1.25;$$

While, if the link between source to relay is not polarized but the link between the relay to destination is polarized, then, we have:

$$C_2^+ = 0.5 + min\{0.5, 1 - 0.5\varepsilon_2\}$$

and

$$C_2^- = 1.$$

As C_K^+ are representative of the most capacity (good channel). Now we examine the appropriateness of these two bounds. If $O \le \varepsilon_2 \le 0.75$, then $C_1^+ \le C_2^+$; and if $O.75 \le \varepsilon_2 \le 1$, then $C_1^+ \le C_2^+$. When $\varepsilon_2 = 0.75$, then all of the C_K^+ s will be equal to each other. It is worth noting that C_0^- and C_0^+ are, for the link between source to relay is polarized and the link between the relay to destination is polarized too, then, we have:

$$C_0^+ = 0.75 + min\{0.75, 1 - 0.5\varepsilon_2\}$$

and

$$C_0^- = 0.75 + min\{0.75, 0.5 + 0.5\varepsilon_2\}$$

Which are always better than two other cases.

Remark1: It is worth-mentioning that by using the polarization we can find more capacity in comparison to the non-polarized case. For example in a point-to-point channel by using channel polarization, the added capacity is:

$$\mathrm{I(X_2; X_1Y_1Y_2) - I(X_2; Y_2) > 0}$$

where X_i is the i-th input and Y_j is the j-th output. In this way, for the proposed channel polarization method of relay channel, the same thing happens. In the other words, According to the capacity region bound of [15], the capacity of the proposed relay channel in Example1 is:

$$C = \min\left\{1 - \frac{\varepsilon_2}{2}, 0.75\right\}$$

which leads $C \leq 0.75$. In fair comparison to the Example 1, the lowest capacity of C_1^+ is 0.75, which shows the extension of the capacity by using polar codes.

Now suppose that W is a binary input relay channel. Let $\{B_n\}_{n\geq 1}$ be an *i.i.d.* uniform random variable valued in {-,+}, with $Pr(B_1 = -) = Pr(B_1 = +) = \frac{1}{2}$ and let us define a relay channel with valued random process $\{W_n: n \geq 0\}$ via

$$W_0 := W, \;\; W_n := W_{n-1}^{B_n}, n \geq 1 \tag{47}$$

Further, we define random processes $\{I_{1\,n}: n \geq 0\}$ and $\{I_{2\,n}: n \geq 0\}$ such that:

$$I_{1\,n} := I_1(W_n), \; I_{2\,n} := I_2(W_n) \tag{48}$$

Lemma 1. The random processes $\{I_1(W_n): n \geq 0\}$ and $\{I_2(W_n): n \geq 0\}$ are bounded martingale.

Proof: Since W_n is a binary input relay channel, $I_1(W_n)$ and $I_2(W_n)$ take values in [0,1]; hence, the mentioned processes are bounded. The martingale property is claimed from (47), (48). The process $(I_1(W_n), I_2(W_n))$ converges almost surely and the limit is

$$(I_{1\infty}, I_{2\infty}) = \lim_{n\to\infty}\left(I_1(W_n), I_2(W_n)\right) \tag{49}$$

Now, the following theorem is given for calculating the Bhattacharya parameter, which is used in relay channels polarization.

Theorem 3. For each given relay channel with $I_1(W) = I_{MAC}$ and $I_2(W) = I_{BC}$ and for the *BC phase*, we have:

$$W_{BC} \to W_{BC}^-, W_{BC}^+: \{Z^-(W) \leq 2Z_{BC}(W), Z^+(W) \leq {Z_{BC}}^2(W)\}, \tag{50}$$

and for *MAC phase* of the relay channel, we have:

$$W_{MAC} \to W_{MAC}^-, W_{MAC}^+: \{Z^-(W) \leq 2Z_{MAC}(W), Z^+(W) \leq {Z_{MAC}}^2(W)\} \tag{51}$$

Proof: We use the fact that for any binary input discrete memory-less channel W, we have:

$$I(W) + Z(W) \geq 1, \tag{52}$$

and

$$I(W)^2 + Z(W)^2 \leq 1 \tag{53}$$

[4]. In polarizing mode, using [3] we get:

$$W^-: Z^-(W) \leq 2Z(W) - Z(W)^2, \tag{54}$$

$$W^+: Z^+(W) = Z(W)^2; \tag{55}$$

and using [5, 7, 15] we can write:

$$Z^-(W) \leq 2Z(W) \tag{56}$$

and

$$Z^+(W) \leq Z(W)^2. \tag{57}$$

Since the relay channel is combined of two *MAC* and *BC* channels, and can be considered as a point-to-point channel. Also, regarding to [16-17], we can write below relations for *MAC* and *BC phase* of relay channel, respectively. For *BC phase*, we have:

$$W_{BC} = \{Z_1^-(W) \leq 2Z_{BC}(W) \text{ for } W_{BC}^- \text{ and } Z_1^+(W) \leq Z_{BC}(W)^2 \text{ for } W_{BC}^+\}; \tag{58}$$

also, for *MAC phase*, we have:

$$W_{MAC} = \{Z_2^-(W) \leq 2Z_{MAC}(W) \text{for } W_{MAC}^- \text{ and } Z_2^+(W) \leq Z_{MAC}(W)^2 \text{ for } W_{MAC}^+\} \tag{59}$$

So, in general, we have:

$$Z(W^-) \leq \max\{Z_1(W^-), Z_2(W^-)\} \leq \max\{2Z_{BC}(W), 2Z_{MAC}(W)\}, \tag{60}$$

$$Z(W^+) \leq \max\{Z_1(W^+), Z_2(W^+)\} \leq \max\{Z_{BC}(W)^2, Z_{MAC}(W)^2\} \tag{61}$$

One can conclude that

$$I({W^-}_{mac}) + I({W^+}_{mac}) \leq 2I(W_{mac}) \tag{62}$$

and

$$I({W^-}_{bc}) + I({W^+}_{bc}) \leq 2I(W_{bc}) \tag{63}$$

for any relay channel that links become polarized, and

$$R \leq \min\{I(W_{bc}), I(W_{mac})\} = 1 - \max\{Z(W_{bc}), Z(W_{mac})\} \tag{64}$$

Although, one can find the capacity by solving an optimization problem like [18], but the polarization technique can increase the capacity as well like [7].

We consider a relay channel with orthogonal receiver component. It consists of three nodes; a source node (S), a relay node (R), and a destination node (D). This system has three directed transmission links: the channel between source and receiver is displayed by W_{SD}, the channel between source and relay is displayed by W_{SR} and the channel between relay and destination is displayed by W_{RD}. We consider all the channel links have been polarized with generator matrix G_n. The relay is stochastically degraded in our system. Therefore, W_{SD} is stochastically degraded with respect to W_{SR}. Hence, if A_{SD} and A_{SR} are good channel sets for polar codes of W_{SD} and W_{SR} channels, respectively [16-17].

Lemma 2. For any two discrete W_{SD} and W_{SR} memory-less channels, if W_{SD} is degraded regard to W_{SR}, then $A_{SD} \subseteq A_{SR}$.

Proof. The proof of this lemma is like the Theorem 1 of [10].

5. Good Index Chosing in Relay Channel

We will always assume that these indices are labeled from 1 to N and that the processing order of the successive decoder is the one implied by this labeling. In this section, we describe our schema for choose a good index to the place set information from between the links which are polarized in the relay channel. First, we represent a polar block of length N by a row vector as in Fig. 2-a, such that any link has block-length $= 2^n$. For N=4, it has been shown that which one of the links are good after polarization. Here we offer a plan for a square screen $N \times N$; This is the similar structure of the Fig. 2-c for showing indices between the polarized link. In the previous sections, we used the word index to refer to one of the synthetic channels which are created by the polarization process. Represents the index in the intersection points of a link in Fig. 2-c that half of them are good for place information bits and remainder are bad, which are proper for frozen bits.

Now consider the relay channel shown Fig. 1. As can be observed, links between SR, SD and RD are determined through W_{SR}, W_{SD}, W_{RD} channels, respectively. For link SD, suppose W_{SD} be a DMC with a binary input X and output Y. Fix a distribution P_x for the random variable X Let $U^{1:N} = X^{1:N}G_N$, where $X^{1:N}$ is a vector of n i.i.d. components are drawn according to P_x. Consider the sets $\mathcal{H}_x$ and $\mathcal{L}_x$ defined as:

$$\mathcal{H}_x = \{i \in [N] : Z(U^i \mid U^{1:i-1}) \geq 1 - 2^{-(N)^\beta}\}, \quad (65)$$

$$\mathcal{L}_x = \{i \in [N] : Z(U^i \mid U^{1:i-1}) \leq 2^{-(N)^\beta}\}. \quad (66)$$

For $i \epsilon \mathcal{H}_x$, the bit U^i is approximately uniformly distributed and independent of $U^{1:i-1}$. In addition, $\mathcal{H}_x$ and $\mathcal{L}_x$ is consisted such that:

$$\lim_{N\to\infty} \frac{|\mathcal{H}_x|}{N} = H(X), \quad (67)$$

$$\lim_{N\to\infty} \frac{|\mathcal{L}_x|}{N} = 1 - H(X). \quad (68)$$

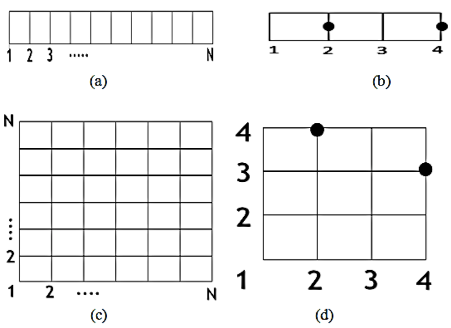

Fig. 2. a: A polar block of length N as a row vector. b: Good channels for polarized channel when N=4. c: A polar block of length $N \times N$ for each link in relay channel. d: A example for shown choose index in a channel with N=4.

Now, assume that the channel output $Y^{1:n}$ is given, and interpret this as side information on $X^{1:n}$. Consider the sets $\mathcal{H}_{X|Y}$ and $\mathcal{L}_{X|Y}$ as below:

$$\mathcal{H}_{X|Y} = \{i \in [N] : Z(U^i \mid U^{1:i-1}, Y^{1:N}) \geq 1 - 2^{-(N)^\beta}\}, \quad (69)$$

$$\mathcal{L}_{X|Y} = \{i \in [N] : Z(U^i \mid U^{1:i-1}, Y^{1:N}) \leq 2^{-(N)^\beta}\}. \quad (70)$$

For $i \epsilon \mathcal{H}_{X|Y}$, U^i is an approximately uniformly distributed and independent of $(U^{1:i-1}, Y^{1:n})$, and for $i \epsilon \mathcal{H}_{X|Y}$, U^i becomes approximately a deterministic function of $(U^{1:i-1}, Y^{1:n})$. Furthermore:

$$\lim_{N\to\infty} \frac{|\mathcal{H}_{X|Y}|}{N} = H(X|Y), \quad (71)$$

$$\lim_{N\to\infty} \frac{|\mathcal{L}_{X|Y}|}{N} = 1 - H(X|Y). \quad (72)$$

To create a polar code for the channel W_{SD}, we proceed now as follows: we place the information in the position indexed by

$$I_{SD} = \mathcal{H}_X \cap \mathcal{L}_{X|Y}. \quad (73)$$

Certainly, if $i \in I_{SD}$, then, U^i is approximately uniformly distributed given $U^{1:i-1}$, since $i \epsilon \mathcal{H}_x$. This implies that, U^i is suitable to contain information. Additionally, U^i given $U^{1:i-1}$ and $Y^{1:n}$, since $i \epsilon \mathcal{L}_{X|Y}$. Using (65)-(68) and the fact that the number of indices in $[N]$ which are neither in $\mathcal{H}_x$ nor in $\mathcal{L}_x$ is $O(N)$, it follows that:

$$\lim_{N\to\infty} \frac{|I_{SD}|}{N} = \lim_{N\to\infty} \frac{|\mathcal{L}_{X|Y} \setminus \mathcal{L}_X|}{N} = \lim_{N\to\infty} \frac{|\mathcal{L}_{X|Y}|}{N} - \lim_{N\to\infty} \frac{|\mathcal{L}_X|}{N} = H(X) - H(X|Y) = I(X;Y) = I_{SD}(W). \quad (74)$$

The remaining positions are frozen. More specifically, they are divided into two subsets, namely and $F_{d-SD} = \mathcal{H}_x^c$, that the frozen indices F_{r-SD} filled with binary bits $F_{r-SD} = \mathcal{H}_x \cap \mathcal{L}_{x|y}^c$, which are chosen uniformly at random and the frozen indices F_{d-SD}, which are chosen according to a deterministic rule. Similarly, to construct a polar code for the channel W_{SR}, we place the information on the positions indexed by:

$$I_{SR} = \mathcal{H}_{X|X_r} \cap \mathcal{L}_{X|X_r,Y_r} \quad (75)$$

and the remaining positions are related to frozen bits. They are divided into two subsets, namely

$$F_{r-SR} = \mathcal{H}_{X|X_r} \cap \mathcal{L}_{X|X_r,Y_r}^{\,c} \quad (76)$$

and

$$F_{d-SR} = \mathcal{H}^c_{X|X_r}, \quad (77)$$

which the frozen indices F_{r-SR} filled with binary bits selected uniformly at random and the frozen indices F_{d-SR} selected based on a deterministic principle. Since:

$$\mathcal{H}_{X|X_r} = \{j \in [N] : Z(U^j \mid U^{1:i-1}, X_r^{1:n}) \geq 1 - 2^{-(N)^\beta}\}, \quad (78)$$

$$\mathcal{L}_{X|X_r} = \{j \in [N] : Z(U^j \mid U^{1:i-1}, X_r^{1:n}) \leq 2^{-(N)^\beta}\}, \quad (79)$$

$$\mathcal{H}_{X|X_r,Y_r} = \{j \in [N] : Z(U^j \,|\, U^{1:i-1}, X_r^{1:n}, Y_r^{1:n}) \geq 1 - 2^{-(N)^\beta}\} \tag{80}$$

$$\mathcal{L}_{X|X_r,Y_r} = \{j \in [N] : Z(U^j \,|\, U^{1:i-1}, X_r^{1:n}, Y_r^{1:n}) \leq 2^{-(N)^\beta}\}. \tag{81}$$

So we have:

$$\lim_{N\to\infty}\frac{|I_{SR}|}{N} = \lim_{N\to\infty}\frac{|\mathcal{L}_{X|X_r,Y_r} \setminus \mathcal{L}_{X|X_r}|}{N} = \lim_{N\to\infty}\frac{|\mathcal{L}_{X|X_r,Y_r}|}{N} - \lim_{N\to\infty}\frac{|\mathcal{L}_{X|X_r}|}{N} = H(X\,|X_r) - H(X\,|X_r,Y_r) = I(X;Y_r\,|X_r) = I_{SR}(W). \tag{82}$$

Finally, for link RD, since

$$V^{1:n} = X_r^{1:n} G_N \, , \tag{83}$$

we place the information on the positions indexed by:

$$I_{RD} = \mathcal{H}_{X_r} \cap \mathcal{L}_{X_r|Y} \tag{84}$$

and the remaining positions are frozen. Based on the previous section, frozen bits related to this links are divided into two subsets F_{r-RD} and F_{d-RD}, which the frozen indices

$$F_{r-RD} = \mathcal{H}_{X_r} \cap \mathcal{L}_{X_r|Y} \tag{85}$$

filled with binary bits chosen uniformly at random and frozen indices

$$F_{d-RD} = \mathcal{H}^c_{X_r} \tag{86}$$

chosen according to a deterministic rule and since:

$$\mathcal{H}_{X_r} = \{k \in [N] : Z(V^k \,|\, V^{1:k-1}) \geq 1 - 2^{-(N)^\beta}\}, \tag{87}$$

$$\mathcal{L}_{X_r} = \{k \in [N] : Z(V^k \,|\, V^{1:k-1}) \leq 2^{-(N)^\beta}\}, \tag{88}$$

$$\mathcal{H}_{X_r|Y} = \{k \in [N] : Z(V^k \,|\, V^{1:k-1}, Y^{1:n}) \geq 1 - 2^{-(N)^\beta}\} \tag{89}$$

$$\mathcal{L}_{X_r|Y} = \{k \in [N] : Z(V^k \,|\, V^{1:k-1}, Y^{1:n}) \leq 2^{-(N)^\beta}\}. \tag{90}$$

So, we have:

$$\lim_{N\to\infty}\frac{|I_{RD}|}{N} = \lim_{N\to\infty}\frac{|\mathcal{L}_{X_r|Y} \setminus \mathcal{L}_{X_r}|}{N} = \lim_{N\to\infty}\frac{|\mathcal{L}_{X_r|Y}|}{N} - \lim_{N\to\infty}\frac{|\mathcal{L}_{X_r}|}{N} = H(X_r) - H(X_r\,|Y) = I(X_r;Y) = I_{RD}(W). \tag{91}$$

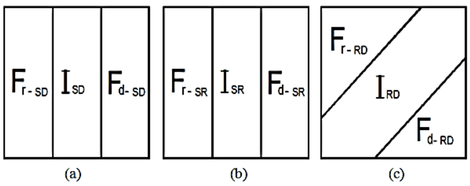

Fig. 3. Graphical representation of the sets associated to the channel coding problem (a. FIF plane for links SD b. FIF plane for links SR c.FIF plane for links RD).

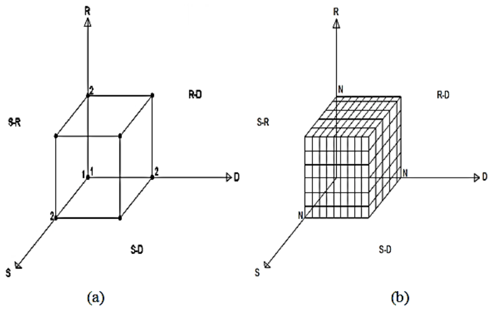

Fig. 4. Graphical representation of set index of relay channel (a. Set indices of relay channel with N=2 and b. Set indices for relay channel with N).

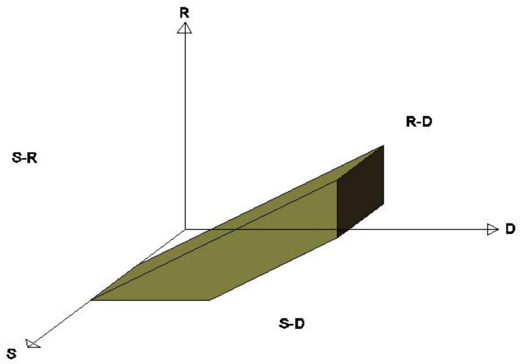

Fig. 5. Graphical representation of good channel in planes FIF

For Fig. 4-b, when $N \longrightarrow \infty$ intersection point become closer and closer to each other and in other words it can be displayed segmentation information as fig.5. When channels are polarize, synthesized channels can be classified into two categories, defining two index sets: the set information bits *I(w,g)* of indices corresponding to good channels and the set frozen bits *F(W)* of indices that belong to bad channels, and also set *F(W)* is consist of two part $F_d(w)$ and $F_r(w)$. For relay channel with *N=2*, it is shown that the set index according to Fig. 4-a., that each point in planes S, R, and D explain the links used in the relay channel. Similarly, Fig. 4-b. depicts relay channel with *N*, that, for example point (*i*,*j*,*k*)=(3,4,4*)* indicate in S-D, S-R, and R-D links 3, 4, and 4 are good respectively. For relay channel that $N \longrightarrow \infty$, set information of good channel expression according to

$$I(w,g) = I_{SD}(w) \cap I_{SR}(w) \cap I_{RD}(w)\, , \tag{92}$$

that shown in Fig. 5. *I(w,g)* represent the set good index for the relay channel polarized.

Now, at the last part of this section, the performance of polar codes for relay channels has been analyzed. The BER performance for this case has been shown when the relay-destination link is a Binary Symmetric Channel (BSC).

In Fig. 6 and Fig. 7, the performance of polar codes for DF and CF relaying in physically degraded relay channel has been shown, respectively. In this way, we utilize the proposed good index choosing based on the Section V. In this analysis, W_{SR} and W_{SD} are independent BSC. The Crossover probabilities for these links are equal to 0.05 and 0.15, respectively.

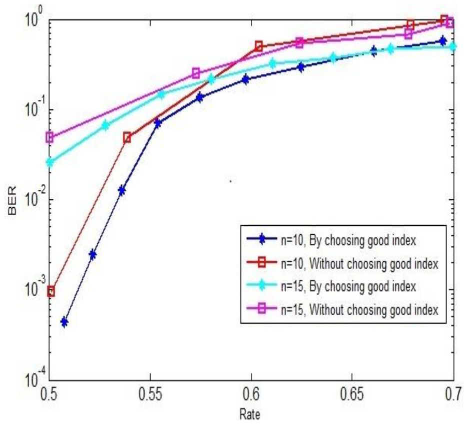

Fig. 6. Comparison of BER performance of polar codes for DF relaying for BSC with the proposed good index choosing method. It has been considered $I(W_{SR}) \approx 0.71, I(W_{RD}) \approx 0.53$ and $I(W_{SD}) \approx 0.31$ in the simulation.

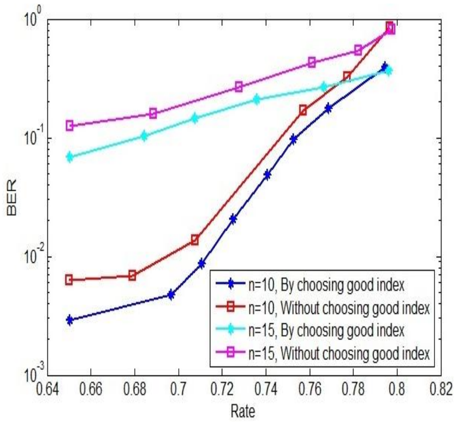

Fig. 7. Comparison of BER performance of polar codes for CF relaying for BSC with the proposed good index choosing method. It has been considered $I(W_{SR}) \approx 0.71, I(W_{RD}) \approx 0.53$ and $I(W_{SD}) \approx 0.31$ in the simulation.

By comparing Fig.6 and Fig.7, one can observe that using the proposed good index choosing method gives lower error probability.

6. conclision

In this paper, we showed that polar codes are suitable for DF and CF relaying with the orthogonal receiver and represent idea about channel polarization specifically for relay channel. It has been considered that for two relays, when the links are polarized, the capacity of one relay increases while the other decreases and the capacity region of the relay changes. It has been shown that polarization make improve the cut-set bound for relay channel. At last, we introduced a new scheme that shows how to choose a good index for polarized relay channels. By using the proposed method, one can find less error probability.References

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